Generating Graphs and Feeling Good Enough

Recap of Alternating + Symmetric Groups

- $G = \text{Sym}(\{1,2,3,4,5\}) = S_5$

  $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix} \in G$ Column notation

  $= (1\ 2\ 5\ 4)(3)$ cycle notation

  $= (1\ 2\ 5\ 4)$
- $id = 1 = (1)(2)(3)(4)(5)$
- $g^{-1} = (1\ 4\ 5\ 2)$
- $h = (3\ 4)(1\ 5) \in G$

  $gh = (1254)(34)(15)$ read left to right

  $= (12)(345)$

- Let $g \in S_n$ with disjoint cycle decomposition $g = c_1 \cdots c_r$

  $g \in A_n \iff$ The number of c_i st $|c_i|$ is even is even
- $g \notin A_5 \quad h \in A_5$

Generation

- From now on let G be a group

- Def^n - $S \subseteq G$ is a generating set for G if all elements of G can be expressed as a product of elements from $S \cup S^{-1}$
- Write $G = \langle S \rangle$

- Example - Let $n \geq 4$

$$
\begin{aligned}
S_n &= \langle \text{all elements of } S_n \rangle & &\text{size } n! \\
&= \langle (12), (13), \ldots, (1n) \rangle & &\text{size } n-1 \\
&= \langle (12), (12, \ldots, n) \rangle & &\text{size } 2
\end{aligned}
$$

$$
\begin{aligned}
A_n &= \langle \text{all elements of } A_n \rangle & &\text{size } \tfrac{n!}{2} \\
&= \langle (123), (124), \ldots, (12n) \rangle & &\text{size } n-2 \\
&= \begin{cases} \langle (123), (12\ldots n) \rangle & \text{if } n \text{ odd} \\ \langle (123), (23\ldots n) \rangle & \text{if } n \text{ even} \end{cases} & &\begin{matrix}\text{size } 2 \\ \text{size } 2\end{matrix}
\end{aligned}
$$

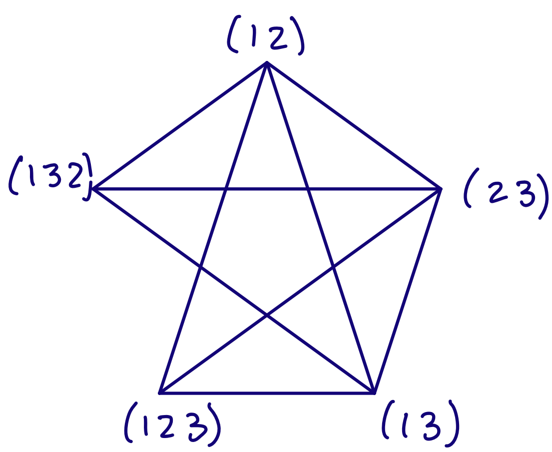
Generating Graphs

- Def^n. The generating graph of G is $\Gamma(G) = (V, E)$ where

$$V = G \setminus \{1\}$$
$$E = \{ (x, y) \in V^2 \mid \langle x, y \rangle = G \}$$

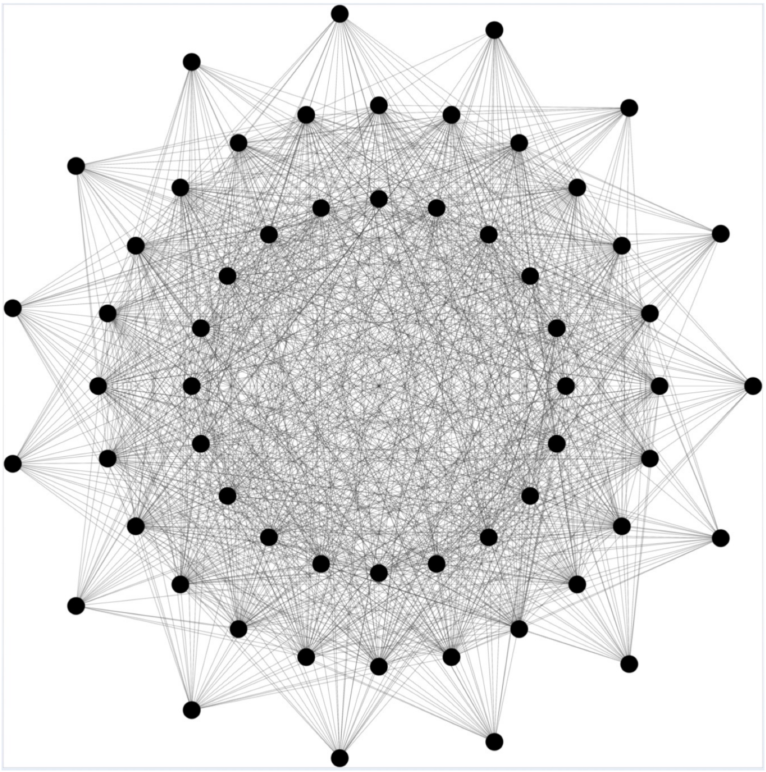
- So $(12) \sim (12\ldots n)$ in $\Gamma(S_n)$
  $(123) \sim (12\ldots n)$ in $\Gamma(A_n)$ for n odd

- Example - $\Gamma(S_3)$



$\langle (132), (123) \rangle \neq S_3$
$\Rightarrow (132) \nsim (123)$

- Example - $\Gamma(A_5)$



This graph was made by Dr Scott Harper

- Def^n - For a graph $\mathcal{G} = (V, E)$
  - $U_1 \subseteq V$ is a clique if $\forall u, v \in V$ $u \neq v$, $(u,v) \in E$
  - $U_2 \subseteq V$ is a coclique if $\forall u, v \in V$, $(u,v) \notin E$

- Example - Let $\mathcal{G} = \Gamma(S_3)$

$U_1 = \{(12), (23), (123)\}$
$U_2 = \{(12), (23), (123), (13)\}$ } cliques

$U_3 = \{(123), (132)\}$ } coclique

- Def^n - A clique (or coclique) is maximal if it is contained in no larger clique (or coclique)

- Example - In $\Gamma(S_3)$

$$U_1 \subsetneq U_2 = V \setminus \{(132)\} \subsetneq V$$

- V is not a clique since $(123) \nsim (132)$
- $U_2 = V \setminus \{(132)\}$ is a clique contained in no larger clique, and so is a maximal clique
- U_1 is contained in U_2 and so is not a maximal clique

- $\forall x \in V \setminus U_3 \quad \exists y \in U_3$ st $x \sim y$
  $\Rightarrow U_3$ is a maximal coclique

Maximal Subgroups

- Let M be a proper subgroup of G.

  $\forall x, y \in M, \quad \langle x, y \rangle \leq M \neq G$.

  Hence $M \setminus \{1\}$ is a coclique in $\Gamma(G)$

- Def^n - M a proper subgroup of G is maximal if there is no group H with $M < H < G$

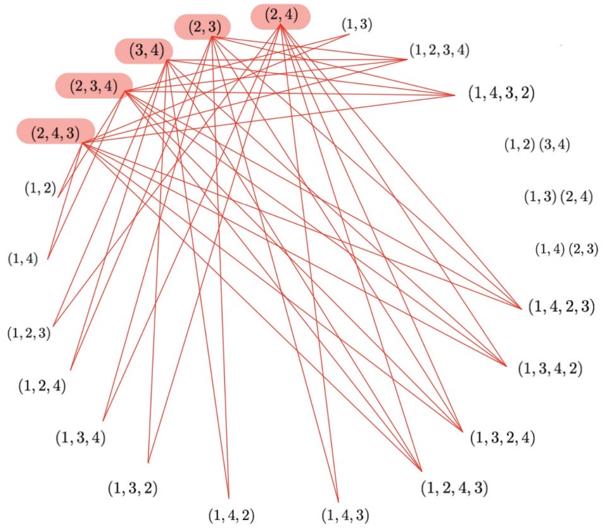
- An equivalent definition:
  $\forall x \in G \setminus M \quad \exists y \in M$ st $\langle x, M \rangle = G$

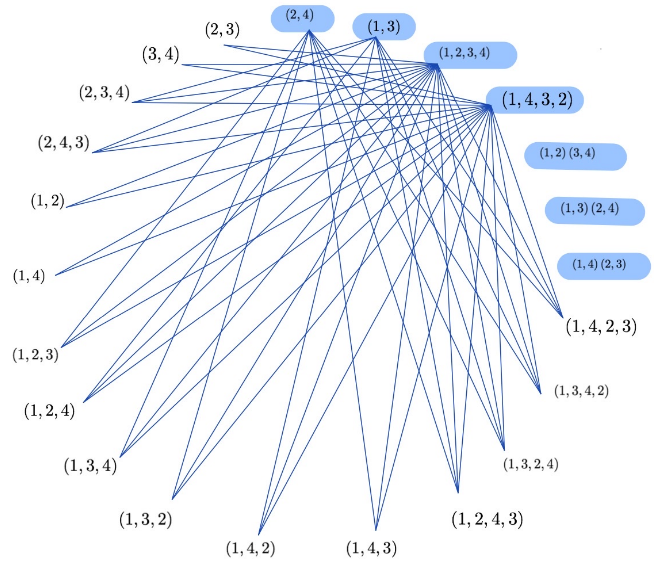
- $M \setminus \{1\}$ is a maximal coclique in $\Gamma(G)$ iff
  $\forall x \in G \setminus M \quad \exists y \in M$ st $\langle x, y \rangle = G$

Similar!

- If M is a max SG of G then $M \setminus \{1\}$ is a coclique in $\Gamma(G)$. Is it a maximal coclique?

- Example





This is a subgraph of $\Gamma(S_4)$ containing all edges with a vertex in $S_3 \leq_{max} S_4$

This is a subgraph of $\Gamma(S_4)$ containing all edges with a vertex in $D \leq_{max} S_4$

$S_3 \setminus \{1\}$ is not a maximal coclique in $\Gamma(S_4)$ since it is contained in the larger coclique $(S_3 \setminus \{1\}) \cup \{(12)(34)\}$

$D \setminus \{1\}$ is a maximal coclique in $\Gamma(S_4)$ since $\forall y \in S_4 \setminus D \ \exists x \in D$ st $\langle x, y \rangle = S_4$

Maximal Subgroups of A_n and S_n $(n \geq 12)$

- A_n is a maximal subgroup of S_n

- Let $1 \leq k < n/2$. Then

$$M_1 = S_k \times S_{n-k} \cong \mathrm{Sym}(\{1, \dots, k\}) \times \mathrm{Sym}(\{k+1, \dots, n\})$$

is a maximal subgroup in M_1.

can shuffle $\{1 \dots k\}$ and $\{k+1, \dots, n\}$ but can't mix points between sets

eg $(12)(3)(45), (123) \in S_3 \times S_2$
$(14), (245) \notin S_3 \times S_2$

- $(S_k \times S_{n-k}) \cap A_n$ is a max SG of A_n

- Let $m, k \geq 2$. Then
  $M_2 = S_k \wr S_m$ is a max SG in S_n

  M can shuffle within the sets
  $\{1 \dots k\}$ $\{k+1 \dots 2k\}$ .... $\{(m-1)k+1, \dots, mk\}$
  and can shuffle the whole sets but can't mix them up

  eg $(1\ 2\ 3 \dots k)(k+1 \dots 2k) \in M_2$ — shuffles within $\{1 \dots k\}$ $\{k+1 \dots 2k\}$

  $(1, k+1, 2, k+1, \dots, k, 2k) \in M_2$ — interchanges as whole sets $\{1 \dots k\}$ $\{k+1 \dots 2k\}$

  $(1, k+1) \notin M_2$ — sends $\{1, 2, \dots, k\}$ to $\{k+1, 2, \dots k\}$ so mixes up

- $(S_k \wr S_m) \cap A_n$ is a maximal SG in A_n

Results

- Let $n \geq 7$, $1 \leq k < n/2$
  (i) If $G = S_n$ and $M = S_k \times S_{n-k}$, then $M \setminus \{1\}$ is a maximal coclique in $\Gamma(G)$ if and only if $\gcd(n,k) = 1$

  (ii) If $G = A_n$ and $M = (S_k \times S_{n-k}) \cap A_n$, then $M \setminus \{1\}$ is a maximal coclique in $\Gamma(G)$

- Let $m,k \geq 2$ and $n = mk \geq 56$
  - (i) If $G = S_n$ and $M = S_k \wr S_m$, then $M \setminus \{1\}$ is a maximal coclique in $\Gamma(G)$
  - (ii) If $G = A_n$ and $M = (S_k \wr S_m) \cap A_n$, then $M \setminus \{1\}$ is a maximal coclique in $\Gamma(G)$

Proof Idea

- We want to show;

  $\forall x \in G \setminus M \quad \exists y \in M \quad st \quad \langle x, y \rangle = G$

  (otherwise $M \cup \{x\}$ would be a larger clique)

- I don't have one argument that works for all possible x so I split into cases

- The good thing about more cases is as you divide further and further you can assume more about x
- The bad thing is now you have loads of cases to deal with!
- Its all about finding the right balance

- General idea;

  For $x \in G \setminus M$ choose $y \in M$ st

  This is a combinatorial property

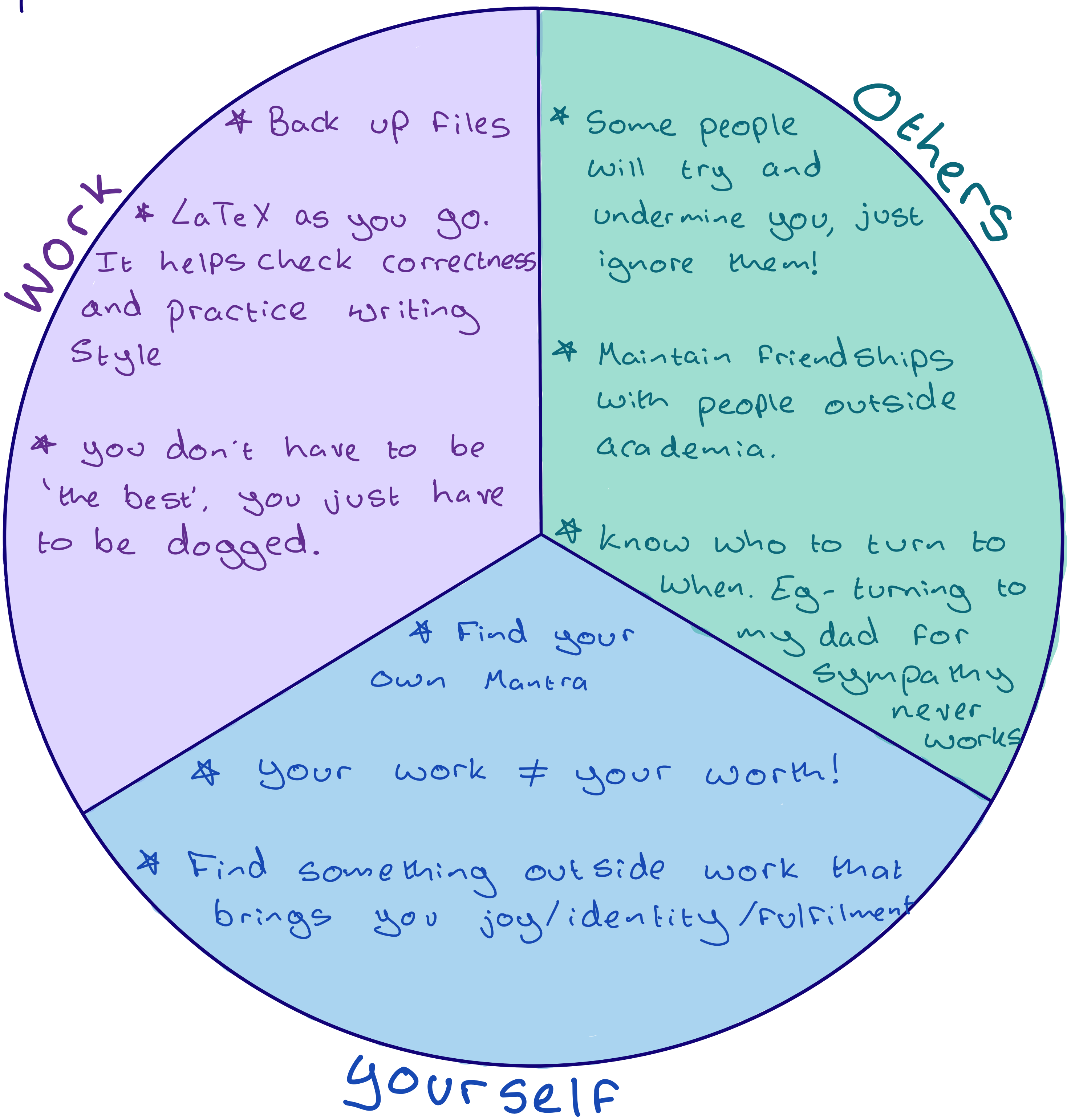
  - (i) $\langle x, y \rangle$ primitive
  - (ii) $\langle x, y \rangle$ contains a 'Jordan element'
  - (iii) $y \in A_n \iff G = A_n$

  relied on proving the existence of primes in certain regions with special properties

- (i) + (ii) $\Rightarrow$ $A_n \leq \langle x, y \rangle$

  (iii) $\Rightarrow$ $G = \langle x, y \rangle$

PhD advice


Work
* Back up files
* LaTeX as you go. It helps check correctness and practice writing style
* you don't have to be 'the best', you just have to be dogged.
Others
* Some people will try and undermine you, just ignore them!
* Maintain friendships with people outside academia.
* know who to turn to when. Eg- turning to my dad for sympathy never works
yourself
* Find your own Mantra
* your work ≠ your worth!
* Find something outside work that brings you joy/identity/fulfilment