The joy of associativity: research and researching in Semigroup Theory

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My story - the facts

- Born and raised in South to a Northern family
- Education always important
- Varied schooling - from experimental infant education to girls' grammar school (a particularly academic kind of high school)
- Bachelor's degree (BA) in Mathematics at the University of York

My story - the facts

- PhD at York
- Lectureships at Bristol, Manchester
- Research Fellowships at TH Darmstadt, Germany, and then York
- 1989 - obtained permanent lectureship in York
- Where to then..?

York and Yorkshire

York Minster

York and Yorkshire

Heslington Hall

York and Yorkshire

Typical Yorkshire scene

Semigroups: the joy of associativity!

Semigroup

A **semigroup** S is a set with an associative binary operation

Usually, we write ab for the product of a and b in S.

The associative law says that for all a, b, c

$$(ab)c = a(bc).$$

Almost every familiar algebraic structure is a semigroup in at least one way.

Monoid

A **monoid** is a semigroup S with a $1 \in S$ such that for all $a \in S$

$$1a = a = a1.$$

Idempotent

An **idempotent** is an element $e \in S$ such that $e = e^2$.

Examples of semigroups and monoids

- Groups
- Multiplicative semigroups of rings, e.g. $M_n(R)$
- **Monoids of transformations** (partial transformations, binary relations) under $\circ$, e.g.

Full transformation monoid $\mathcal{T}_X$

$$\mathcal{T}_X : \; = \; \{\alpha \mid \alpha : X \to X\}$$

- The **endomorphism monoid** End(A) for any algebra A.
- The **free semigroup** X^+ and the **free monoid** X^* on any set X.

What is special/different about semigroups?

- Semigroups are represented by **functions** and not by **bijections**.
- Many techniques of 'classical' algebra do not work - they rely upon having inverses.
- For example, to obtain a quotient/homomorphic image H of a group G we start with a normal subgroup N. We define $a \sim b$ iff $b^{-1}a \in N$. Then $\sim$ is an equivalence relation (in fact, a congruence). Further, $N = [1]$ and $[g] = gN$.
  Then $H =: G/N = \{gN : g \in G\}$ and is a group under $gNhN = ghN$.
- For semigroups, we need to know each congruence class, not just the congruence class of a particular element. We need to consider congruences **directly**.
- Semigroups are the most ubiquitous class of algebras such that the techniques of universal algebra are essential. But, they have their own special tools and techniques.

Being a female mathematician/algebraist/semigroup theorist

- The UK has a culture of boys doing science (including mathematics) and girls studying the arts and social sciences.
- This is changing, if slowly. Only 6% of UK maths professors are female.
- The pipeline is being primed, but is ‘leaky’. However, understanding is emerging that unless science and mathematics wants to attract only single minded, single people, it needs to be friendlier to women and families.
- As a young mathematician I concentrated on publishing papers, teaching well whatever anyone asked me to teach, and getting my next job. I was helped by some wonderful mentors: Fountain, Howie, Munn, Bulman-Fleming and Márki. Now, my collaborative circle is very supportive.
- The playing field is not level. Academia has become commercialised and competitive.

What can be done?

- Realisation that 'something has to be done'.
- The London Mathematical Society, and the Royal Society, both have had initiatives to help women in mathematics and science. Many other countries are following suit.
- Schemes such as Piscopia provide support and routes to mentoring.
- The world is becoming a more technical place, and suddenly the realisation is setting in that mathematicians rule the world.
- Gradually quality over quantity is being appreciated.
- Find good mentors and take advice as broadly as you can.
- To young mathematicians I would say, follow your dreams and take the opportunities open where they are, and where they are not, do your best to change things for the next generation.

Regularity properties of graph products

The construction of a **graph product** is essentially multiplicative - hence in the framework of **semigroup theory**. It allows us to glue together semigroups/monoids/groups to produce new ones.

What **algebraic properties** do graph products have, depending on the same properties of the ingredients?

Fountain and Kambites (2009)

The graph product of right cancellative monoids is right cancellative.

A semigroup S is **right cancellative** if for all $a, b, c \in S$

$$ba = ca \Rightarrow b = c.$$

Regular semigroups

Regularity

A semigroup S is **regular** if for all $a \in S$ there exists a $b \in S$ such that

$$a = aba.$$

If $a = aba$, then immediately

$$(ab)^2 = (ab)(ab) = (aba)b = ab,$$

so that $ab \in E(S) = \{e : e = e^2\}$.

Many natural examples of semigroups are regular (such as $\mathcal{T}_X$), but...many are not, such as X^+ and X^*.

We will see that graph products are not, in general, regular.

Presentations - semigroup presentations

Let X be a non-empty set. The **free semigroup** X^+ consists of all finite sequences

$$x_1 \circ x_2 \circ \cdots \circ x_n, \quad n \in \mathbb{N}, x_i \in X$$

with operation $\circ$ given by

$$(x_1 \circ x_2 \circ \cdots \circ x_n) \circ (y_1 \circ y_2 \circ \cdots \circ y_m) = x_1 \circ x_2 \circ \cdots \circ x_n \circ y_1 \circ y_2 \circ \cdots \circ y_m.$$

An **identity** is an expression $u = v$ where $u, v \in X^+$

$$\text{e.g. } x \circ y = y \circ x.$$

R denotes a **set of identities**.

Presentations - semigroup presentations

A presentation

for a semigroup S looks like: $\langle X \mid R \rangle$.

The elements of S are written as sequences of elements of X

Two sequences of elements of X are **equal in** S if and only if you can get from one to the other by a series of steps

$$c \circ u \circ d \leftrightarrow c \circ v \circ d$$

where $u = v$ is a relation. e.g. if $x \circ y = y \circ x \in R$, then **in** S

$$w \circ x \circ y \circ x \circ v = w \circ x^2 \circ y \circ v.$$

Graph products of semigroups are given by a presentation, as are graph products of monoids, groups, inverse semigroups.

The importance of graph products

Graph products include:

- Graph groups and monoids, also known as right-angled Artin groups/monoids, free partially commutative groups/monoids, trace groups(!)/monoids.
  (Finitely generated monoids and groups defined by presentations in which the only relations have forms $a \circ b = b \circ a$ for generators a, b.)
- Free products of semigroups/monoids/groups.
- Restricted direct products of semigroups/monoids/groups.

Graph products of semigroups

Let $\Gamma = \Gamma(V, E)$ be a simple and undirected graph with no loops.

Let

$$\mathcal{S} = \{S_\alpha : \alpha \in V\}$$

be a set of mutually disjoint semigroups, called **vertex semigroups**.

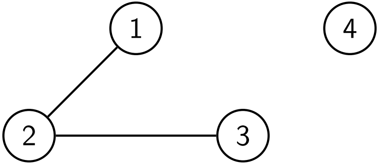
The **graph product**

$\mathscr{G}\mathscr{P} = \mathscr{G}\mathscr{P}(\Gamma, \mathcal{S})$ of $\mathcal{S}$ with respect to Γ is defined by the presentation

$$\mathscr{G}\mathscr{P} = \langle X \mid R \rangle$$

where $X = \bigcup_{\alpha \in V} S_\alpha$ and with defining relations R:
(R_1) $x \circ y = xy$ $(x, y \in S_\alpha, \alpha \in V)$;
(R_2) $x \circ y = y \circ x$ $(x \in S_\alpha, y \in S_\beta, (\alpha, \beta) \in E)$.

What does this mean?



In the corresponding $\mathscr{G}\mathscr{P}$ how do we handle (with natural convention for labelling)

$$s_1 \circ s_2 \circ s_4 \circ s_2' \circ s_3 \circ s_2''?$$

We can write this as

$$s_1 \circ s_2 \circ s_4 \circ (s_2' \circ s_3) \circ s_2'' \to s_1 \circ s_2 \circ s_4 \circ (s_3 \circ s_2') \circ s_2''$$

and then as

$$s_1 \circ s_2 \circ s_4 \circ s_3 \circ (s_2' \circ s_2'') \to s_1 \circ s_2 \circ s_4 \circ s_3 \circ s_2' s_2''.$$

Note that $s_1 \circ s_4$ is not regular as $s_1 \circ s_4 \neq s_1 \circ s_4 \circ w \circ s_1 \circ s_4$.

Left Foata normal form

We show that elements in $\mathscr{GP}$ may be written in a normal form we refer to as **left Foata normal form.**

In the example above,

$$(s_1 \circ s_2) \circ s_4 \circ (s_3 \circ s_2' s_2'')$$

is in left Foata normal form.

Regularity, Abundancy and Fountainicity

It is a **fact** that a semigroup S is regular if and only if for every $a \in S$ we have $e, f \in E(S)$ such that

$$e \,\mathcal{R}\, a \,\mathcal{L}\, f,$$

where here $\mathcal{R}$ and $\mathcal{L}$ are Green's relations of mutual divisibility.

If we replace $\mathcal{R}, \mathcal{L}$ by larger relations $\mathcal{R}^*, \mathcal{L}^*$, or $\widetilde{\mathcal{R}}, \widetilde{\mathcal{L}}$ what if we insist on having idempotents in these classes?

Definition

A semigroup S is **abundant** if every $\mathcal{R}^*$-class and every $\mathcal{L}^*$-class contains an idempotent.
Weak abundancy or **Fountainicity** is defined analogously, using $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{L}}$.

Regular $\Rightarrow$ abundant $\Rightarrow$ Fountain

Regularity, abundancy and Fountainicity

Examples Regular semigroups, cancellative monoids, $M_n(\mathbb{Z})$, restriction monoids, partial transformation monoids, $\cdots$

These ideas arise from many sources and have many names.

Abundancy originally from notions of projectivity for acts; weak abundancy from various studies of small ordered categories, some in the context of Ehresmann's work on pseudo-groups.

See the work of Fountain, Lawson, Cockett, Manes, Jackson, Stokes, El Qallali, Gomes, Szendrei and many others.

Fountainicity and abundancy, the results.

Alqahtani, Gould and Yang (2020)

Any graph product of abundant (Fountain) semigroups is abundant (Fountain).

Gould and Yang (2021)

Any graph product of abundant (Fountain) monoids is abundant (Fountain).

The proofs are **very** different for monoids, since an extra relation (identifying the monoid identities!) is involved, which increases the complexity of the reductions.

Applications and extensions

Corollary

Any free product of abundant (Fountain) semigroups or monoids is abundant (Fountain).

Corollary

Any restricted direct product of abundant (Fountain) semigroups or monoids is abundant (Fountain).

Corollary: Fountain, Kambites, 2009

Any graph product of right cancellative monoids is right cancellative.

A key message to everyone

Thank you for listening.

Any questions :-)