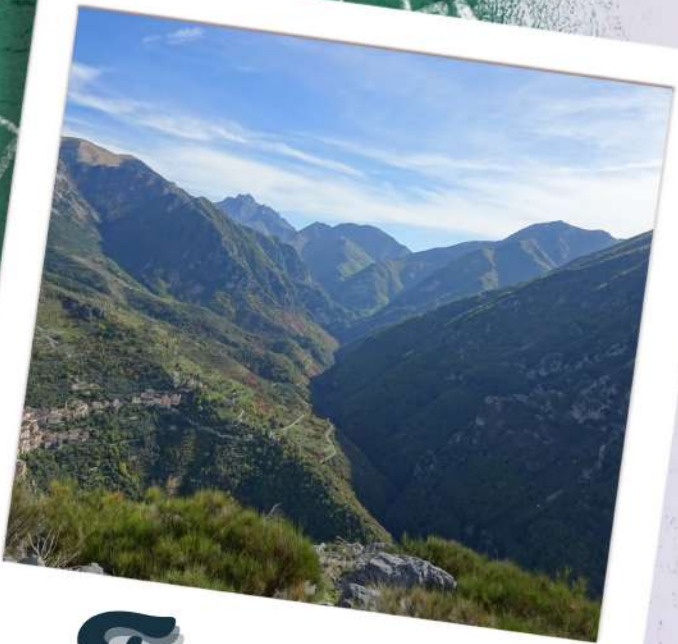
LIANA HEUBERGER

research associate in algebraic geometry

WORKING WITH ALASTAIR CRAW

LOVES: yoga, hiking, poster making, the feeling of having a new article on the arXiv feed

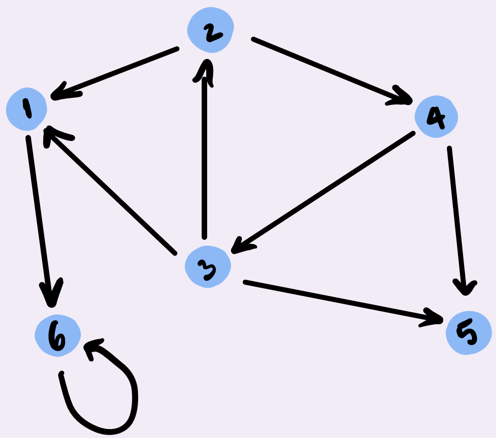
HATES: stereotypes about pure maths, queueing at the 4W cafe, having fussed over this font for 40 minutes

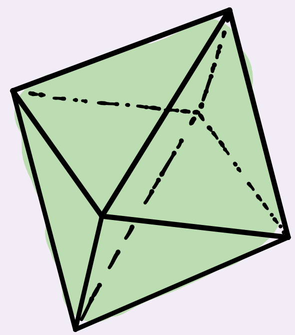


Graphs, groups and shapes

Liana Heuberger

University of Bath





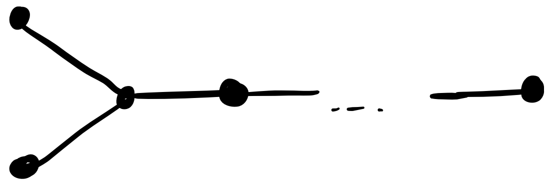
PiWORKS Seminar 2024

Main objects of interest:

A_n
$n \geq 1$

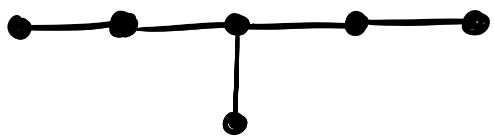
(n nodes)

D_n
$n \geq 4$

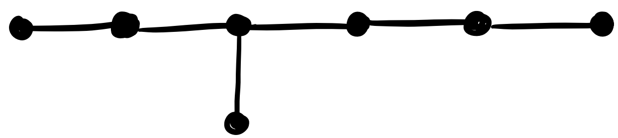


(n nodes)

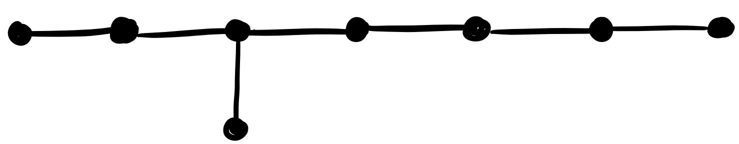
E_6



E_7



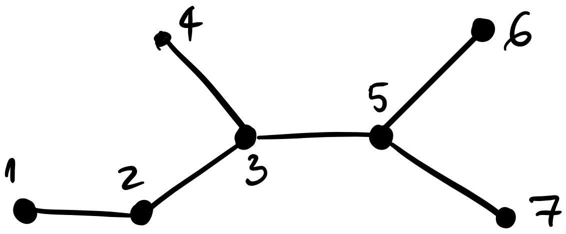
E_8



They are the answer to many a classification problem in various areas.

GRAPHS

G = (non-oriented) graph w/ no loops or multi edges



$$A_G = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\wr$$

$$D_G = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{pmatrix}$$

Form its ADJACENCY MATRIX $A_G = (a_{ij})_{i,j \text{ vertices}}$

where $a_{ij} = \begin{cases} 1 & \text{if } \exists \text{ edge } i \bullet\!\!-\!\!-\!\!\bullet j \\ 0 & \text{otherwise.} \end{cases}$

QUESTION: Given $\alpha \in \mathbb{R}$, can we classify graphs such that the largest eigenvalue of A_G is $< \alpha$?

(Recall α is an **eigenvalue** of A_G if $A_G \cdot v = \alpha \cdot v$ for some vector v.
Such a v is an **eigenvector** of A_G.)

ANSWER: For $\alpha = 2$, these are the ADE diagrams. Indeed:

PROPOSITION: Let G be a connected graph. TFAE:

(1) The largest eigenvalue of A_G is < 2.

(2) G is of type ADE.

this is called the spectral radius of G, $\rho(G)$

$(2) \Rightarrow (1)$ can compute that $\rho(ADE) < 2$.

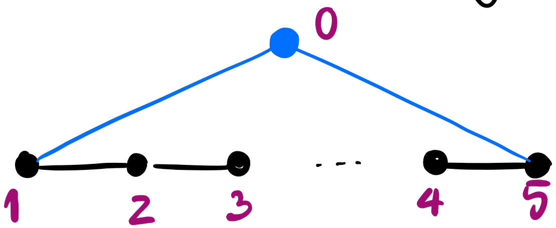
$(1) \Rightarrow (2)$ we use the following

Fact: If G' is a connected subgraph of G, then
$$\rho(G') \leq \rho(G).$$

Trick: extend the diagrams:

$\tilde{A}_n$

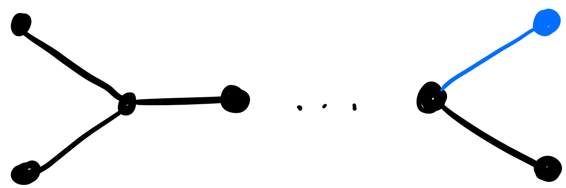
$n \geq 1$



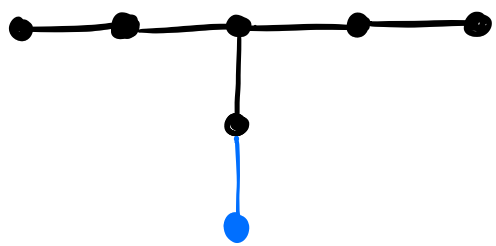
$$A_{\tilde{A}_n} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & & & \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & & & 0 & 1 \\ 1 & 0 & & & 1 & 0 \end{pmatrix}}_{n+1} \begin{pmatrix} ? \end{pmatrix} = 2 \begin{pmatrix} ? \end{pmatrix}$$

$\tilde{D}_n$

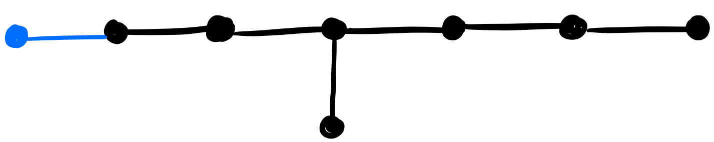
$n \geq 4$



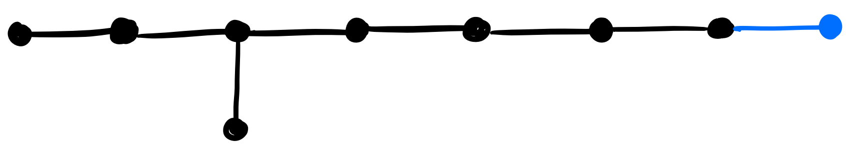
$\tilde{E}_6$



$\tilde{E}_7$



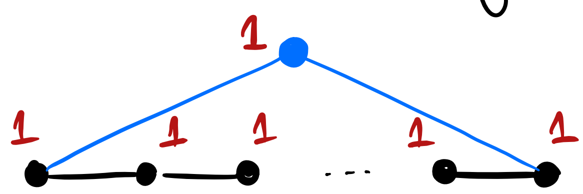
$\tilde{E}_8$



Claim: $\rho(\widetilde{ADE}) = 2$

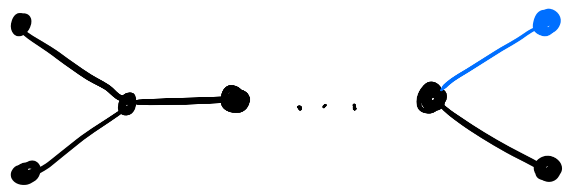
TRICK: extend the diagrams:

$\tilde{A}_n$, $n \geq 1$



$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & & & \\ \vdots & \vdots & & \ddots & & \\ 0 & 0 & & & 0 & 1 \\ 1 & 0 & & & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}$$

$\tilde{D}_n$, $n \geq 4$



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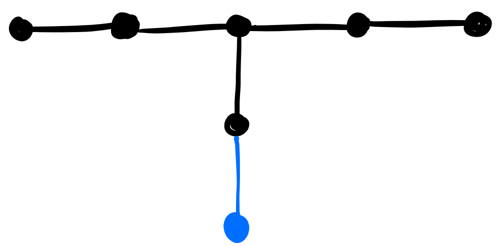

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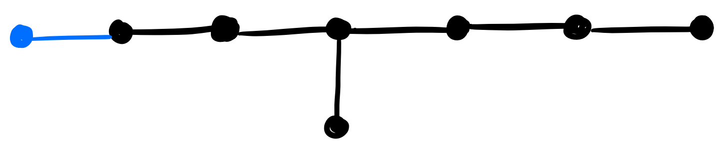
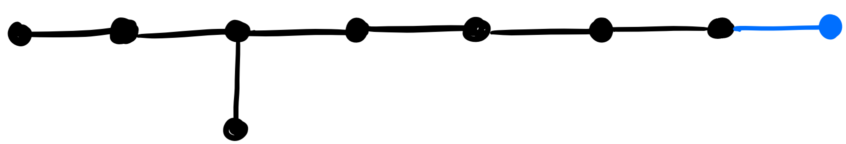


$\tilde{E}_8$



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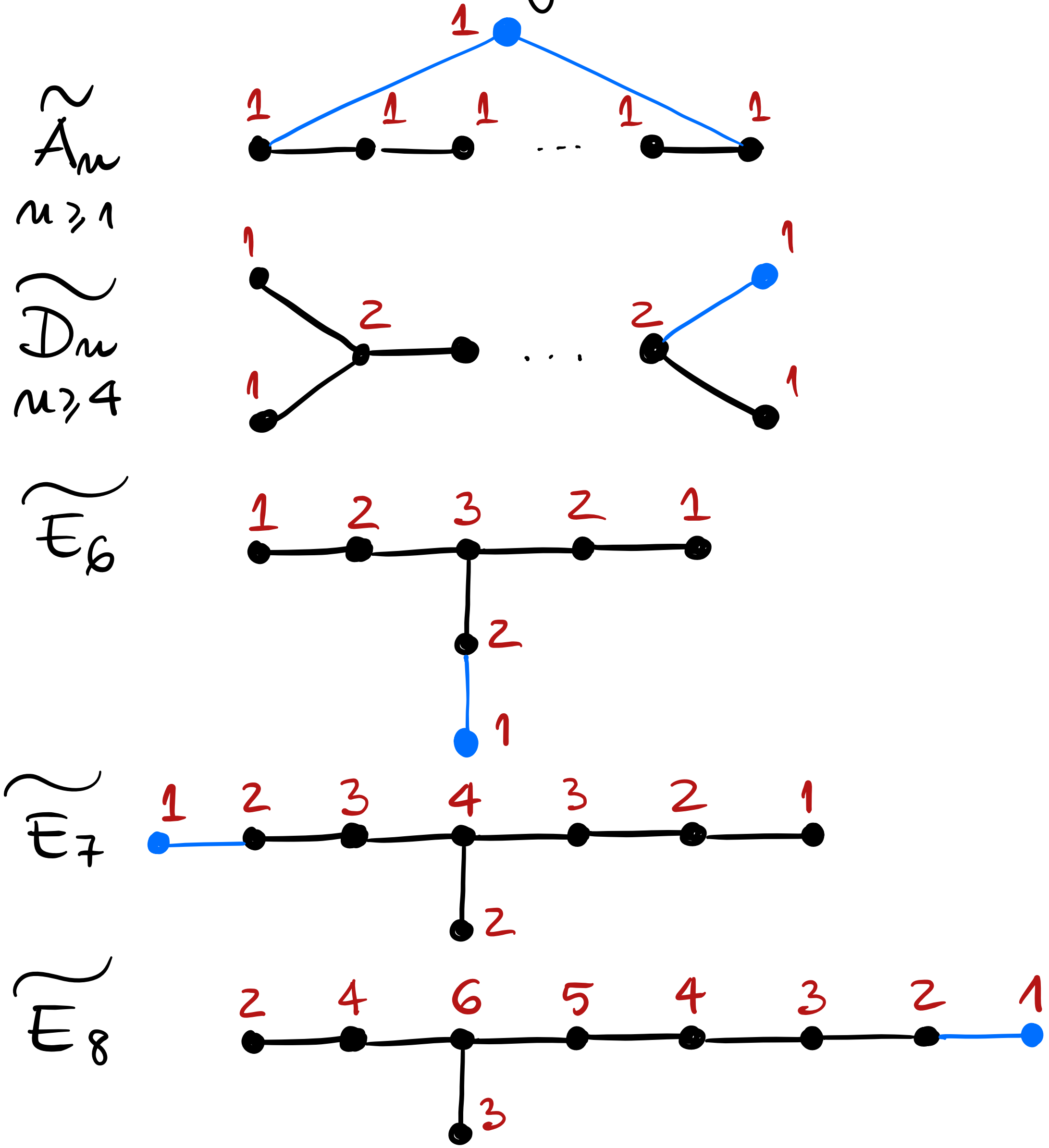

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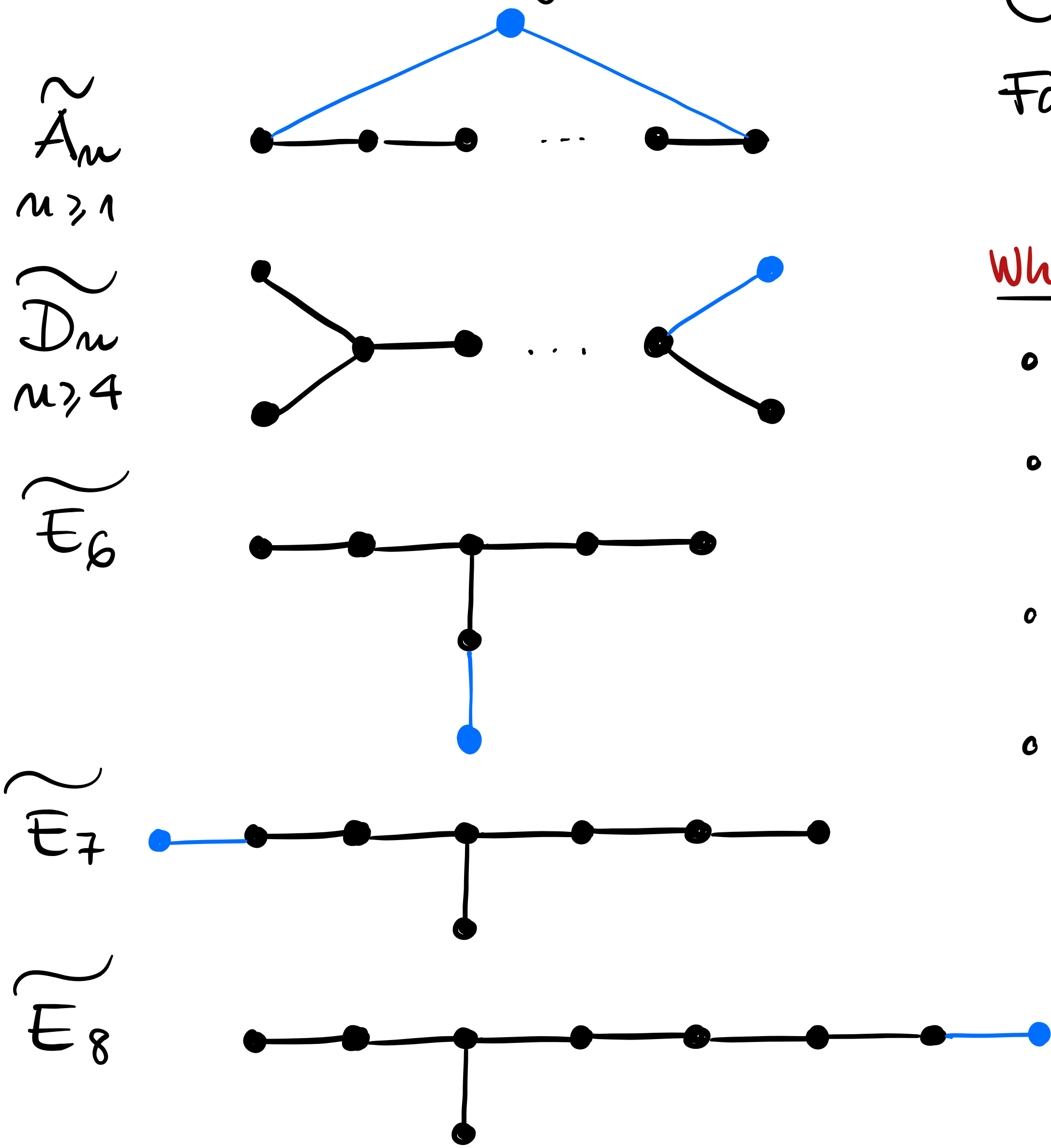

Claim: $\rho(\widetilde{ADE}) = 2$

Claim + Fact $\Longrightarrow$ Any G with $\rho(G) < 2$ can't contain $\widetilde{ADE}$.

Who is G?

- $\tilde{A}_n \not\subset G \Longrightarrow G$ is a tree.
- $\tilde{D}_4 \not\subset G \longrightarrow G$ has at most trivalent vertices
- $\tilde{D}_n \not\subset G \longrightarrow G$ has at most ONE trivalent vertex
- $\tilde{E}_k \not\subset G \longrightarrow$ restrictions on lengths of arms

ONLY A_n D_n $E_{6,7,8}$

SURVIVE.

Second incarnation:

FINITE SUBGROUPS OF $SL(2,\mathbb{C})$.

$$SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{C} \\ ad-bc=1 \end{array} \right\}$$

Recall a GROUP Γ is a set with an operation "$\cdot$" such that

0. $\forall A, B \in \Gamma$: $A \cdot B \in \Gamma$
1. $\forall A, B, C \in \Gamma$: $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ ASSOCIATIVITY
2. $\exists E \in \Gamma$ st $\forall A \in \Gamma$: $A \cdot E = E \cdot A = A$ IDENTITY ELEMENT
3. $\forall A \in \Gamma$ $\exists B \in \Gamma$ st $A \cdot B = B \cdot A = E$ INVERSE ELEMENT.

Proposition: Up to conjugation, a finite subgroup of $SL(2,\mathbb{C})$ is one of:

(A_n) A cyclic group of order $n+1$ generated by $\left\langle \begin{pmatrix} \varepsilon_{n+1} & 0 \\ 0 & \varepsilon_{n+1}^{-1} \end{pmatrix} \right\rangle$

(D_n) A binary dihedral group of order $4(n+2)$: $\left\langle \begin{pmatrix} \varepsilon_{2(n-2)} & 0 \\ 0 & \varepsilon_{2(n-2)}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$

(E_6) The binary tetrahedral group :

$$\left\langle \begin{pmatrix} \varepsilon_4 & 0 \\ 0 & \varepsilon_4^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} \varepsilon_8^7 & \varepsilon_8^7 \\ \varepsilon_8^5 & \varepsilon_8 \end{pmatrix} \right\rangle$$

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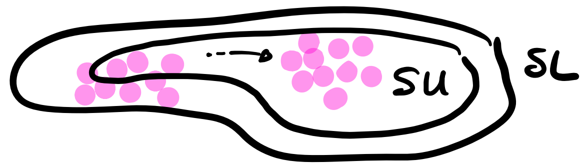
$$\left\langle \begin{pmatrix} \varepsilon_5^3 & 0 \\ 0 & \varepsilon_5^2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} -\varepsilon_5 + \varepsilon_5^4 & \varepsilon_5^2 - \varepsilon_5^3 \\ \varepsilon_5^2 - \varepsilon_5^3 & \varepsilon_5 - \varepsilon_5^4 \end{pmatrix} \right\rangle$$

where $\varepsilon_k = e^{2\pi i/k}$, in other words $(\varepsilon_k)^k = 1$.

<u>Idea</u> : $SU(2) \leq SL(2,\mathbb{C})$ maximal compact subgroup.

$\left\{ \begin{array}{l} A \cdot A^* = I_2 \\ \det A = 1 \end{array} \right\}$ $\{\det A = 1\}$ $\Longrightarrow$ any finite sgr of $SL(2,\mathbb{C})$ can be conjugated into $SU(2)$.

[$\forall S < SL(2,\mathbb{C})$ finite $\exists A \in SL(2,\mathbb{C})$ s.t. $ASA^{-1} \in SU(2)$.]



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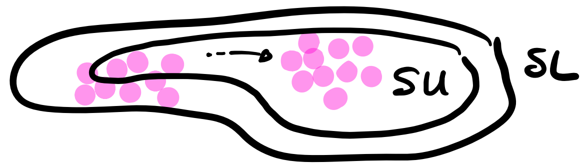
$\left[\forall \underset{\text{finite}}{S} < SL(2,\mathbb{C}) \ \exists \ A \in SL(2,\mathbb{C}) \text{ s.t. } ASA^{-1} \in SU(2) \right]$



$\exists$ a 2:1 cover

$\Big\downarrow$

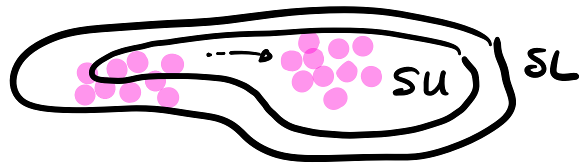
$SO(3)$ $\rightsquigarrow$ can lift the classification from $SO(3)$ to $SL(2,\mathbb{C})$

$\left\{ \begin{matrix} A \cdot {}^{T}\!A = I_3 \\ \det A = 1 \end{matrix} \right\}$

Idea: $SU(2) \leq SL(2,\mathbb{C})$ maximal compact subgroup.

$SU(2) = \left\{ \begin{array}{l} A \cdot A^* = I_2 \\ \det A = 1 \end{array} \right\}$, $SL(2,\mathbb{C}) = \{ \det A = 1 \}$ $\Longrightarrow$ any finite sgr of $SL(2,\mathbb{C})$ can be conjugated into $SU(2)$.

$$\left[\forall \underset{\text{finite}}{S} < SL(2,\mathbb{C}) \ \exists \ A \in SL(2,\mathbb{C}) \text{ st. } ASA^{-1} \in SU(2) \right]$$



$$SU(2) \xrightarrow{\exists \text{ a } 2:1 \text{ cover}} SO(3)$$

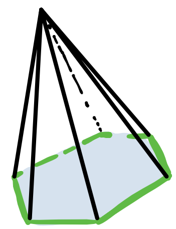
$SO(3) = \left\{ \begin{array}{l} A \cdot A^T = I_3 \\ \det A = 1 \end{array} \right\}$

$SO(3) \Longrightarrow$ can lift the classification from $SO(3)$ to $SL(2,\mathbb{C})$

$SO(3)$ has well-known finite subgroups:

these are the symmetry groups of objects in $\mathbb{R}^3$!

A_n: Cone over a regular $(n+1)$-gon



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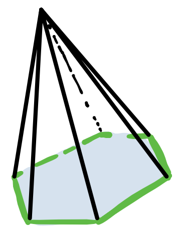
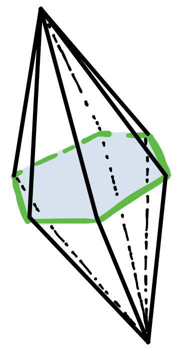


D_n: Double cone over a regular $2(n-2)$-gon

$n \geq 4$



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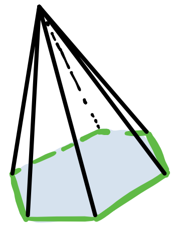

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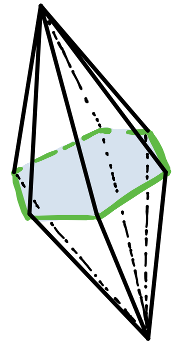
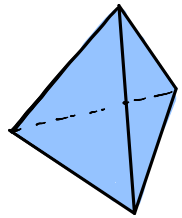
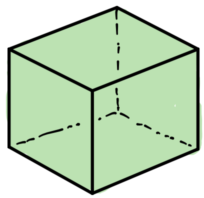


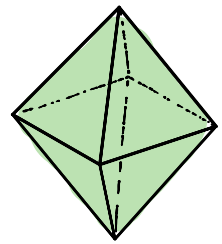
E_6: Regular tetrahedron



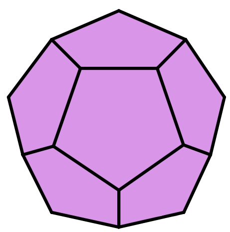
E_7: Cube



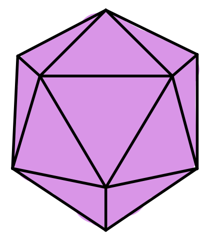
or octahedron



E_8: Dodecahedron



or icosahedron



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where $\varepsilon_k = e^{2\pi i/k}$, in other words $(\varepsilon_k)^k = 1$.

Why are we labelling these subgroups w/ A, D & E ?

- Each group has a finite number of irreducible representations

REP FACTS

1. a representation of a finite group Γ is a group homomorphism $\rho : \Gamma \to GL(V)$ for some Vector space V.
2. $\dim(\rho) = \dim(V)$ dimension / degree
3. an irreducible representation (or iRREP) is one that admits no non-trivial subrepresentations.
4. operations : $\oplus$, $\otimes$.
5. let $\rho_0 \dots \rho_n$ be the irreps of Γ.

any representation $V = \bigoplus_{i=0}^{n} \rho_i^{\otimes a_i} =: \bigoplus_{i=0}^{n} a_i \cdot \rho_i$

Running example: $D_5 = \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;\middle|\; \varepsilon^6 = 1 \text{ primitive} \right\rangle$.

$\boxed{\text{THIS IS ALREADY A REPRESENTATION}}$

DENOTE BY V.

6 irreps $\rho_0 \dots \rho_5$

- Can form its McKay quiver (an oriented graph)

VERTICES: $\rho_0 \dots \rho_5$

ARROWS: $\rho_i \to \rho_j$

$a_{ij} \in \mathbb{N}$ where

$$V \otimes \rho_i = \bigoplus_j a_{ij} \rho_j$$

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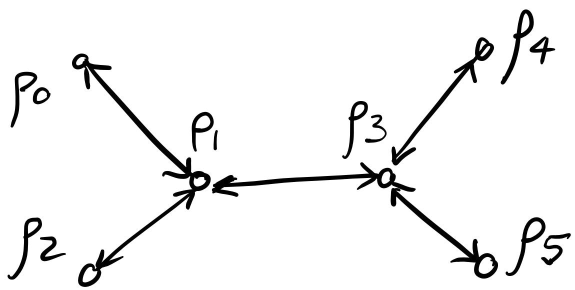
In the case of D_5:



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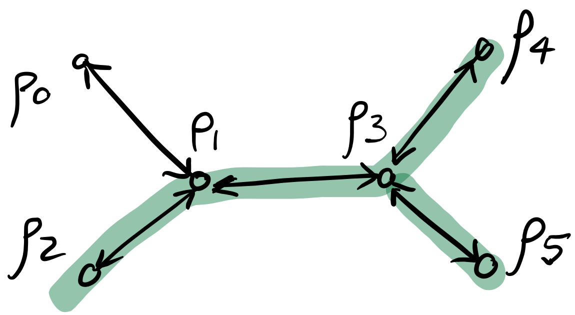

Remark: When we ignore the trivial representation, we obtain exactly the graph D_5 from the first slide!

The same is true for the rest of the As, Ds & Es.

Geometric POV: Each subgroup of $SL(2,\mathbb{C})$ acts on $\mathbb{C}^2$ by multiplication.

We form the quotient $\mathbb{C}^2/\Gamma$, which is an affine algebraic surface.

$\hookrightarrow$ We can write down its equation!

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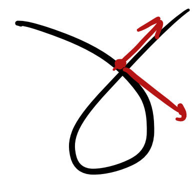
We form the quotient $\mathbb{C}^2/\Gamma$, which is an affine algebraic surface.

↳ We can write down its equation!

Ex: D_5 leads to the hypersurface singularity given by:

$$(x^2 + y^2 z + z^4 = 0) \subset \mathbb{C}^3.$$

→ hard to study!



Geometric POV: Each subgroup of $SL(2,\mathbb{C})$ acts on $\mathbb{C}^2$ by multiplication.

We form the quotient $\mathbb{C}^2/\Gamma$, which is an affine algebraic surface.

→ We can write down its equation!

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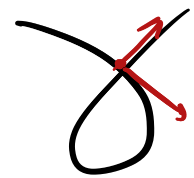
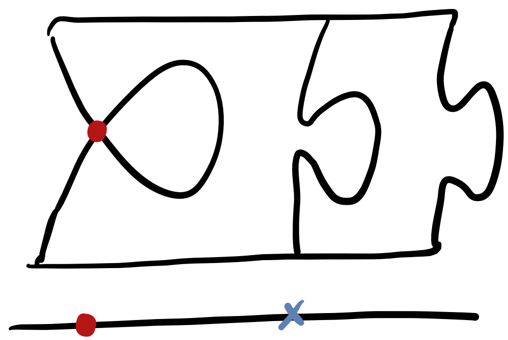


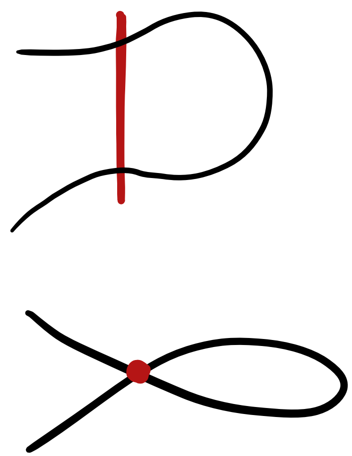
There are two ways to get rid of such points in geometry:

DEFORMATION

- Harder to control geom. properties of fibres
- resulting space may not be compact



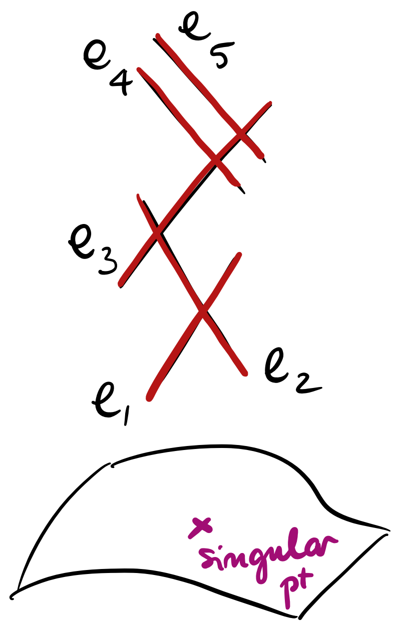
BLOWING UP



- the curve itself becomes smooth
- the space it lives inside is more complicated

In the case of D_5:

Its minimal resolution Y looks like:



Where each $e_i \simeq \mathbb{P}^1$ (projective line)

We call e_i EXCEPTIONAL.

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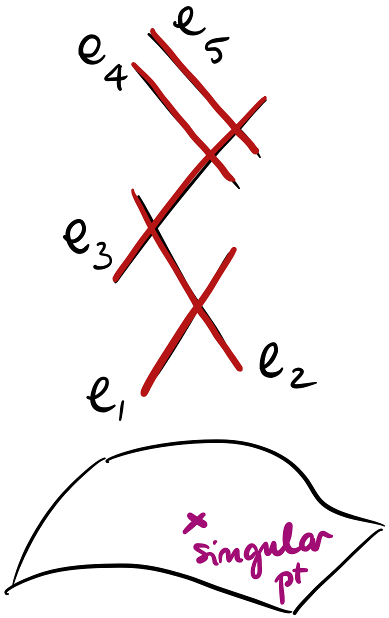

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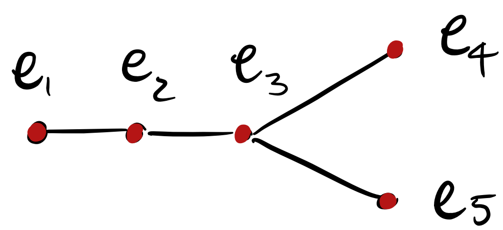

AGAIN THE D5 GRAPH!

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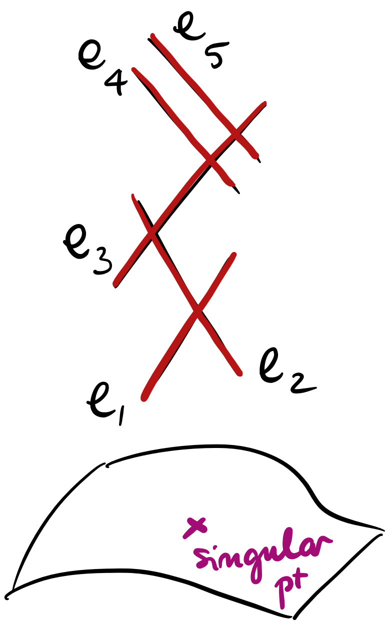

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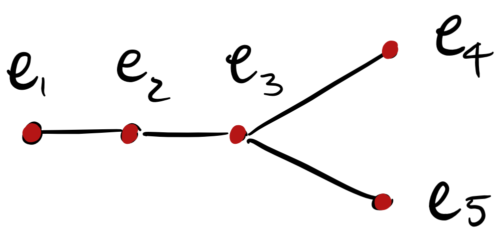

AGAIN THE D5 GRAPH!

PUNCHLINE: This bijection is called the McKay CORRESPONDANCE and it holds for all $\Gamma \subset SL(2,\mathbb{C})$ finite:

$$\left\{ \rho_i \text{ nontrivial irrep of } \Gamma \right\} \xleftrightarrow{1:1} \left\{ \text{exceptional divisors of } Y \longrightarrow \mathbb{C}^2/\Gamma \right\}$$

ALGEBRA   GEOMETRY

Beyond $SL(2,\mathbb{C})$?

DIMENSION 3. Let $\Gamma \subset SL(3,\mathbb{C})$ be finite.

Example: $\Gamma = \frac{1}{6}(1,2,3) = \left\{ \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^3 \end{pmatrix} \middle| \varepsilon^6 = 1 \right\} \subset SL(3,\mathbb{C})$.

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In particular, $V = \rho_1 \oplus \rho_2 \oplus \rho_3$

$$\rho_i \otimes \rho_j = \varepsilon^{i+j} = \rho_{i+j \,(\mathrm{mod}\, 6)}$$

The McKay quiver has 6 vertices $\rho_0 \dots \rho_5$. Arrows:

$$V\otimes\rho_i = (\rho_1\oplus\rho_2\oplus\rho_3)\otimes\rho_i = \underset{(\mathrm{mod}\,6)}{\rho_{i+1}} \oplus \underset{(\mathrm{mod}\,6)}{\rho_{i+2}} \oplus \underset{(\mathrm{mod}\,6)}{\rho_{i+3}} = \sum_{j=0}^{5} a_{ij}\rho_j$$

$$\begin{tikzcd}
 & \cdots & & & & & \\
\cdots \arrow[r] & \rho_1 \arrow[r] \arrow[d] & \rho_2 \arrow[r] \arrow[d] & \rho_3 \arrow[r] \arrow[d] & \rho_4 \arrow[r] \arrow[d] & \rho_5 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \rho_4 \arrow[r] \arrow[d] \arrow[ul] & \rho_5 \arrow[r] \arrow[d] \arrow[ul] & \rho_0 \arrow[r] \arrow[d] \arrow[ul] & \rho_1 \arrow[r] \arrow[d] \arrow[ul] & \rho_2 \arrow[r] \arrow[d] \arrow[ul] & \cdots \arrow[ul] \\
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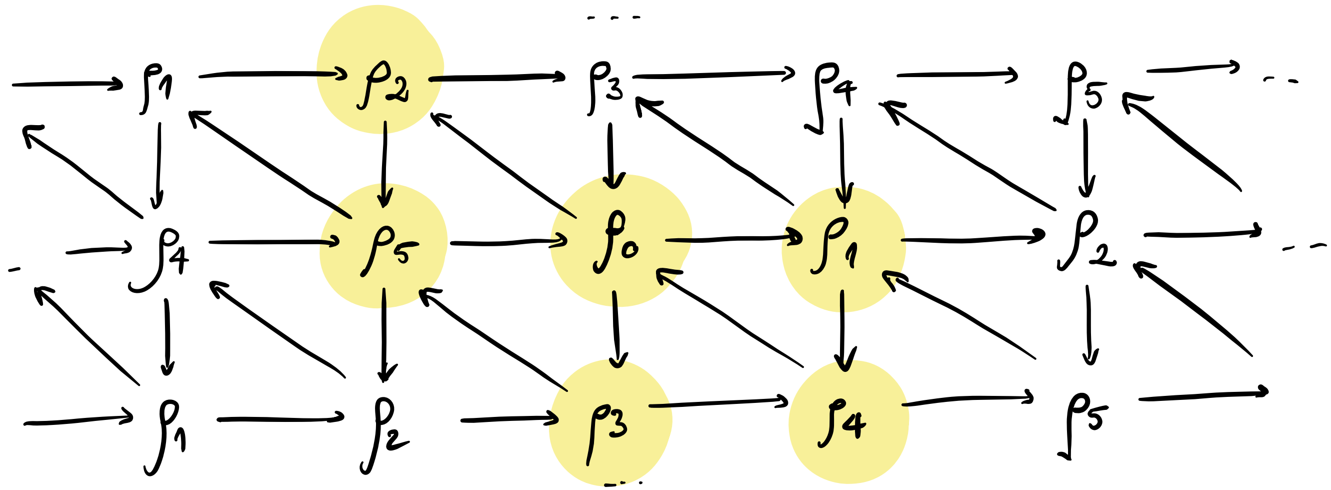
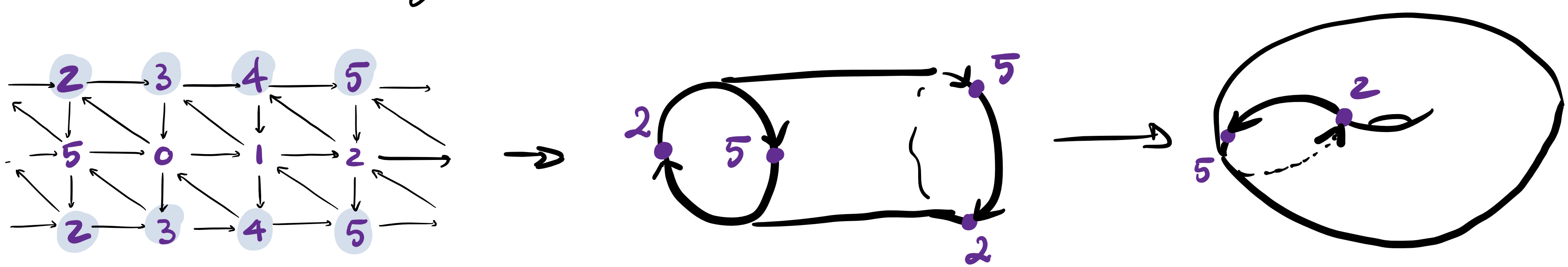


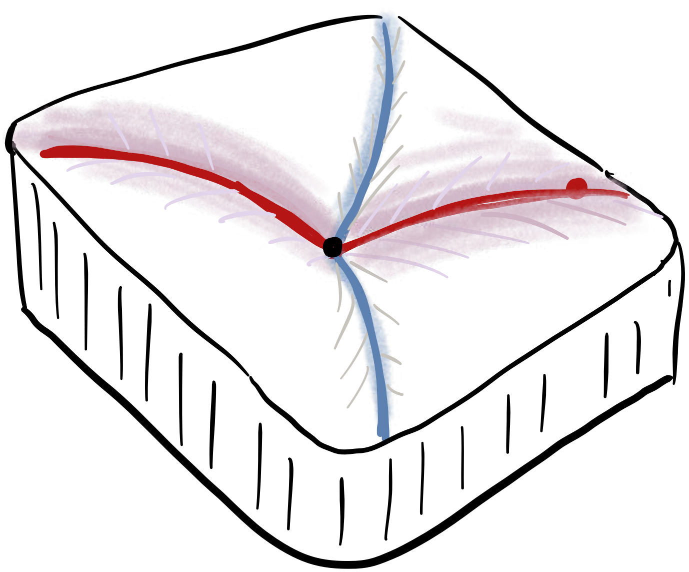
NOTE: you want every vertex to appear just ONCE, so this quiver naturally lives on a torus.



What about the geometry side? Want to study the variety.

$$X = \mathbb{C}^3/\Gamma \quad (\text{for } \Gamma \text{ in the ex})$$

Again, you can write its equations and study the singularities:



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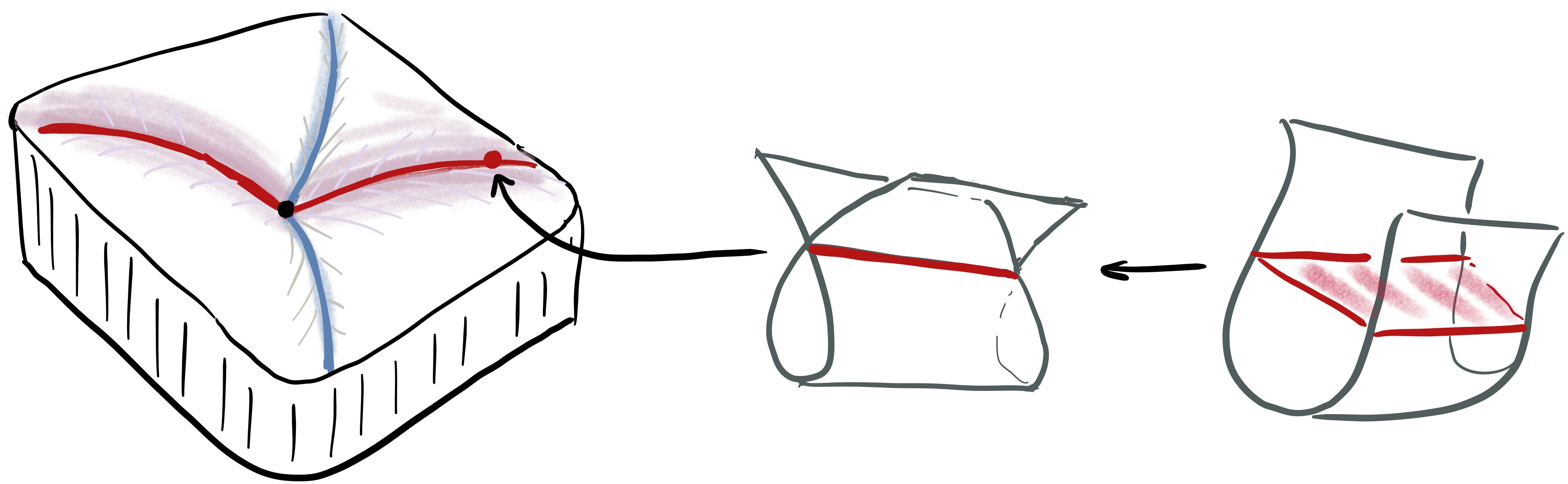

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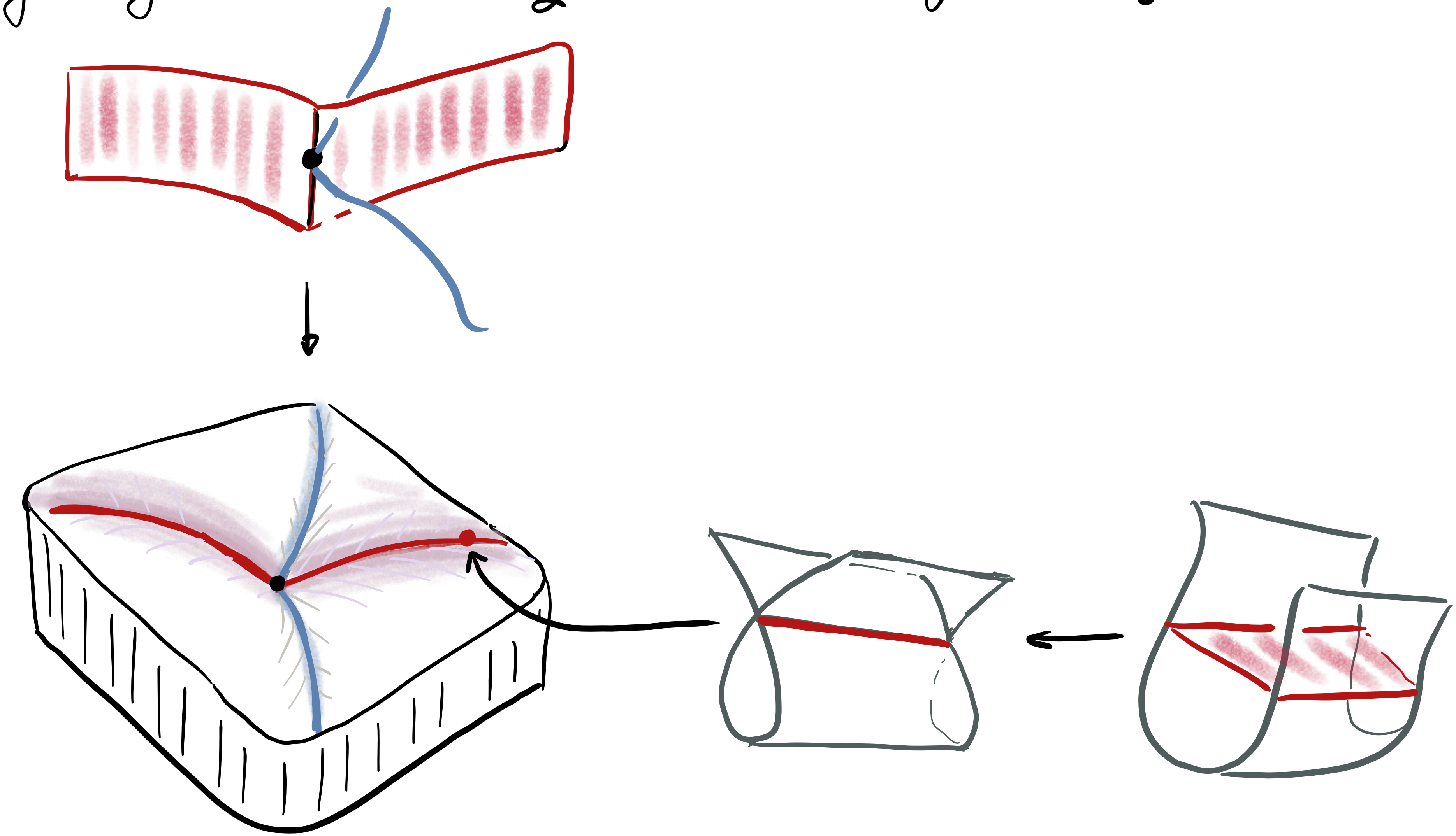

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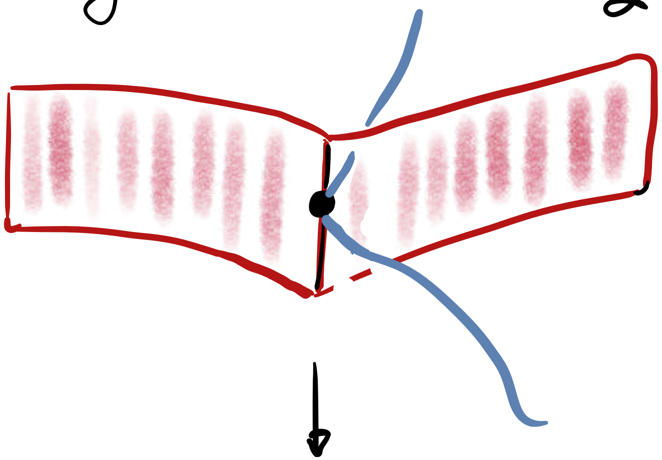
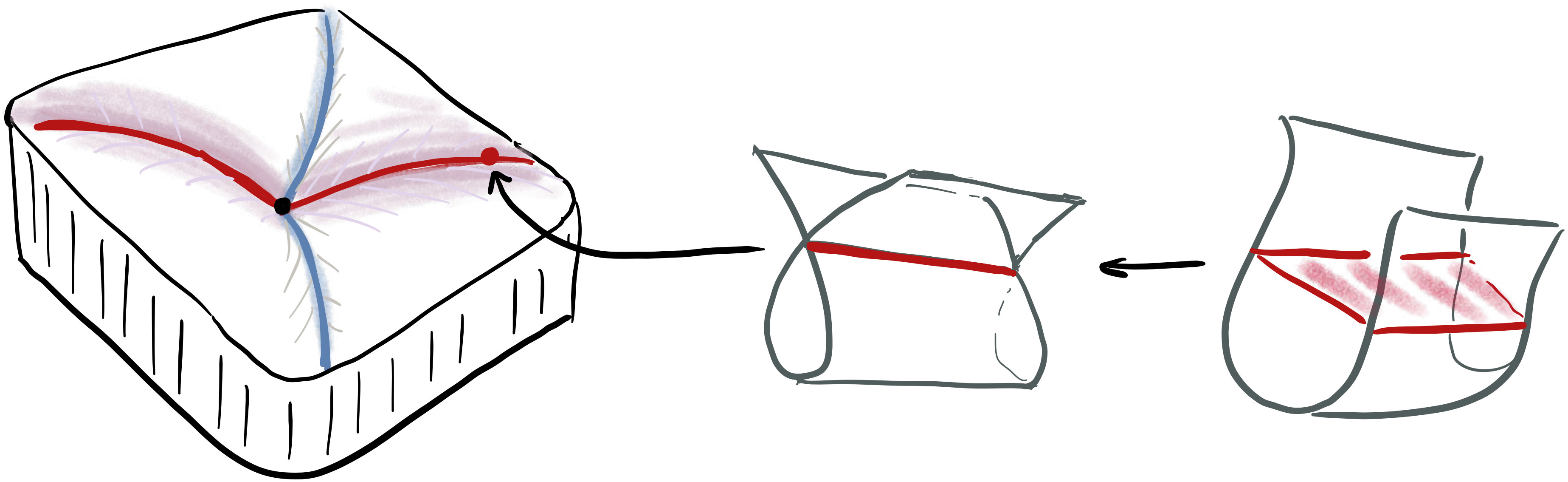


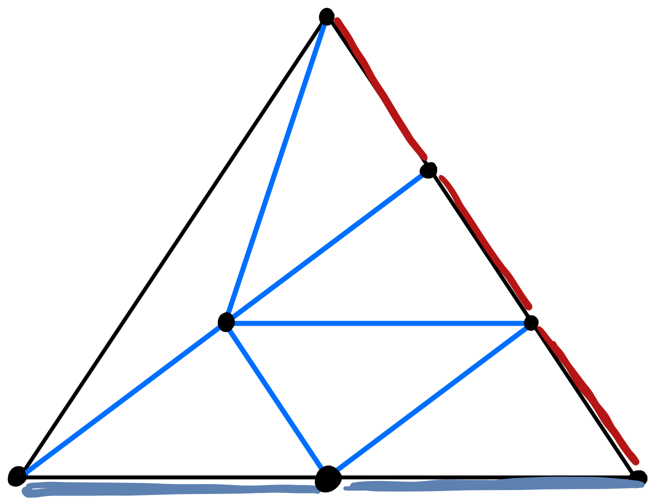
need more steps, but most confusingly:

The resolution is not unique!



BUT: There exists a special resolution Y for which the McKay correspondence holds!

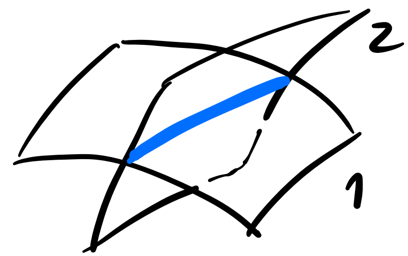
We can still represent it as a dual graph of sorts.



• = surfaces

— = curves

1 • — • 2 = surface 1 • intersects surface 2 • along the curve —



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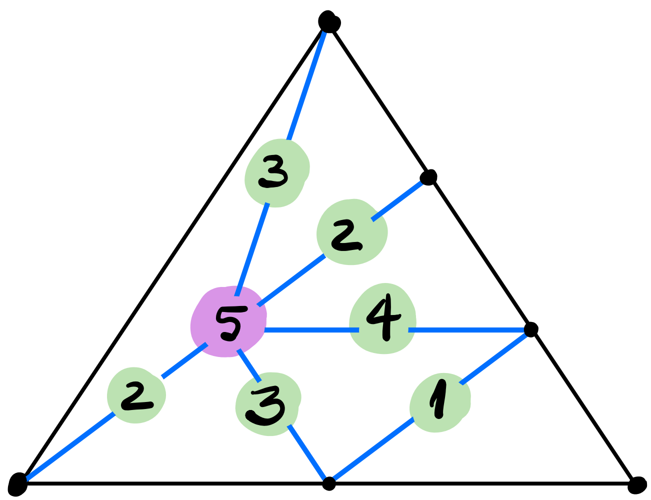

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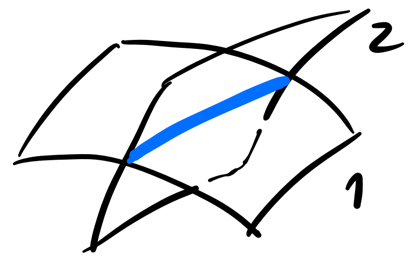
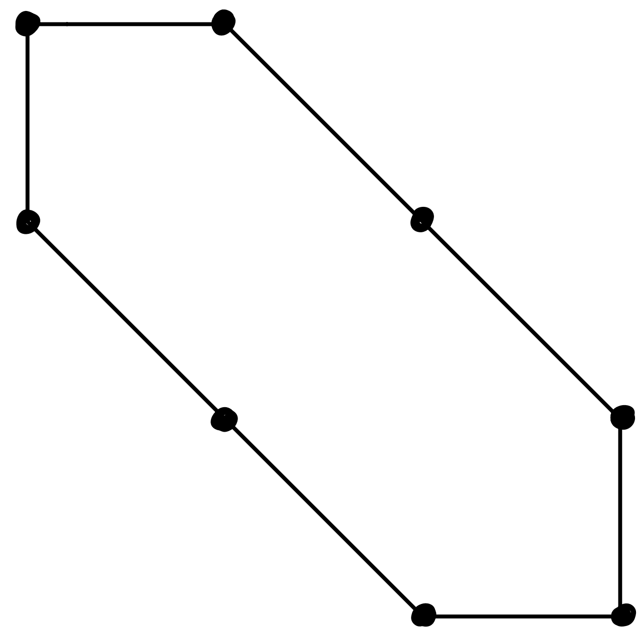


We can associate $\rho_1, \rho_2, \dots, \rho_5$ to every compact exceptional curve and surface.

$\Updownarrow$

strictly internal

Beyond $X = \mathbb{C}^3/\Gamma$ – start from an arbitrary polygon:



$$X = \mathbb{C}^{\#\text{vertices}} /\!\!/ (\mathbb{C}^*)^k \times \text{finite group.} = \mathbb{C}^6 /\!\!/ (\mathbb{C}^*)^3 \times \Gamma$$

Still an algebraic variety.

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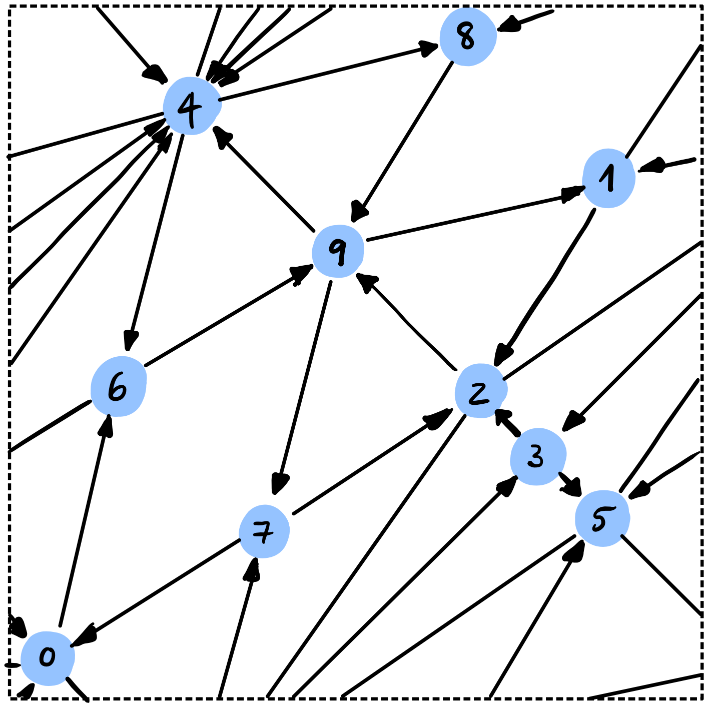
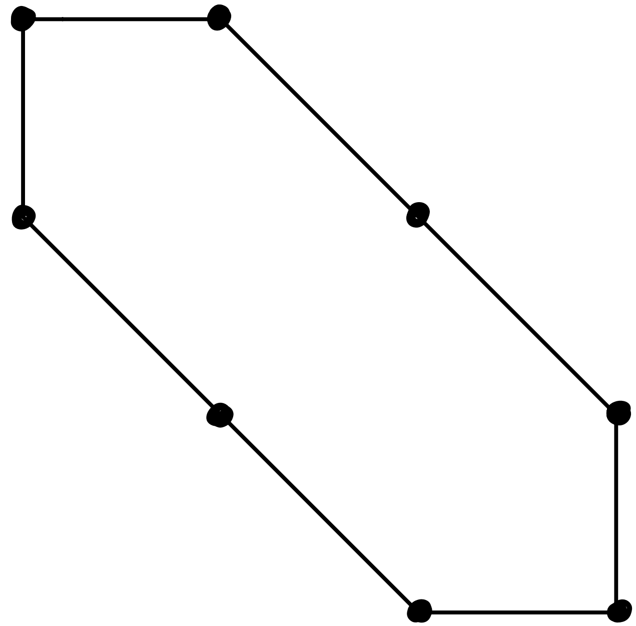


$\longleftrightarrow$



Build a quiver
and a resolution

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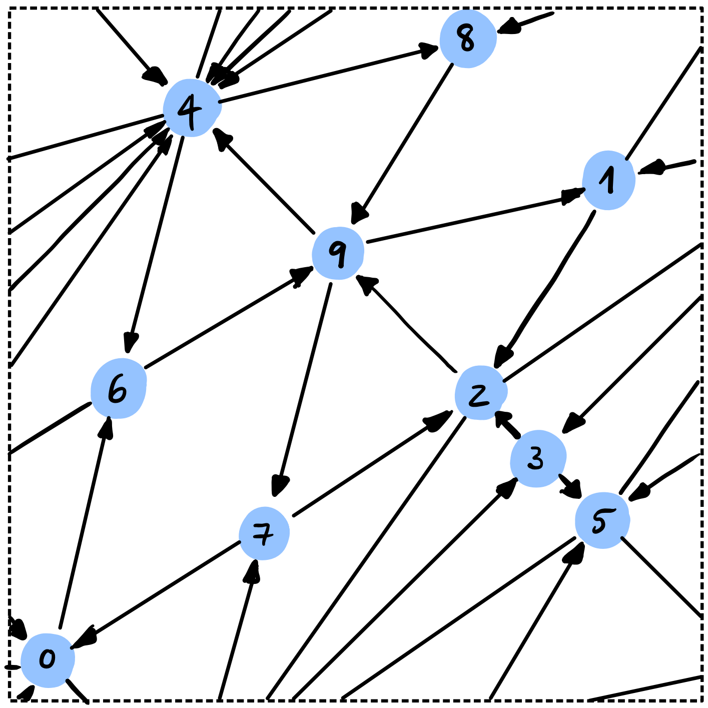

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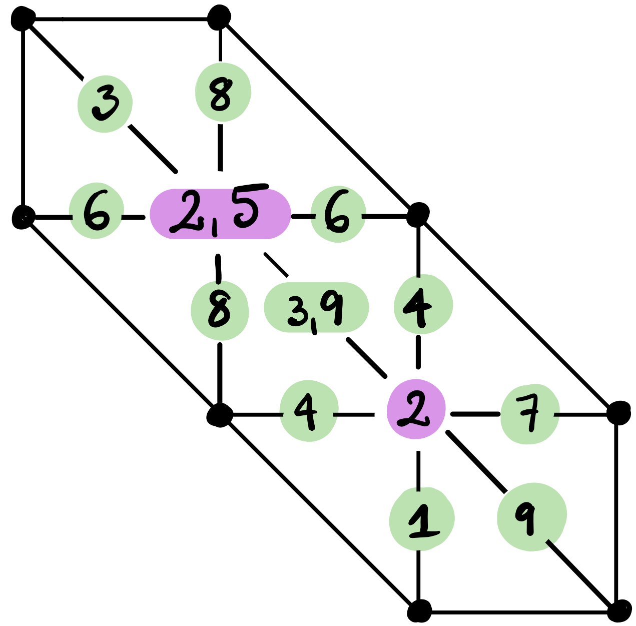


Build a quiver and a resolution, then establish the correspondence!

$$X = \mathbb{C}^{\#\text{vertices}} /\!\!/ (\mathbb{C}^*)^k \times \text{finite group} = \mathbb{C}^6 /\!\!/ (\mathbb{C}^*)^3 \times \Gamma$$

Still an algebraic variety.

What's new to this case?

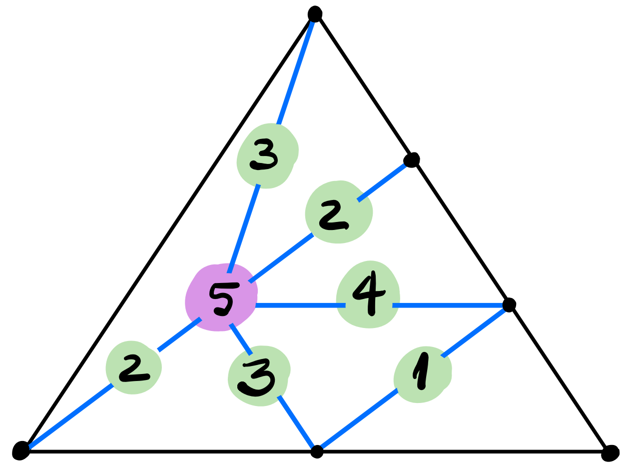
1. Interior lattice points can be marked w/ the same irrep (eg 2)

2. Interior line segments can be marked w/ more than one irrep (eg 3 & 9)

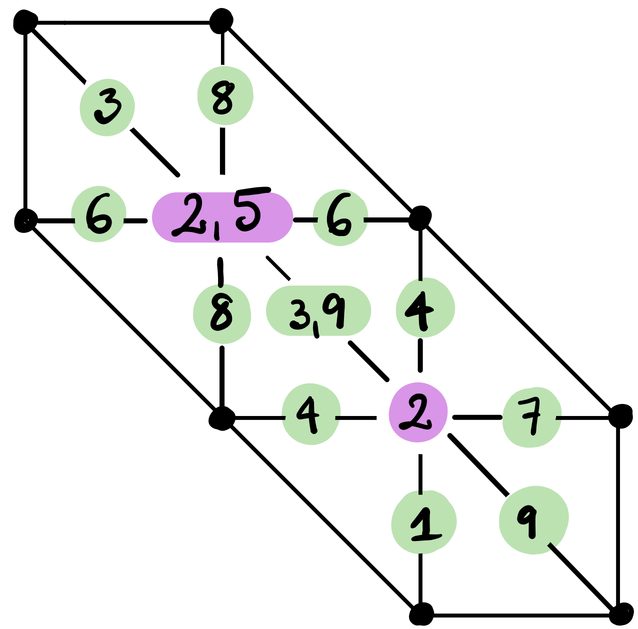
3. The marking of an interior line segment is not determined by the hyperplane containing it (eg 3 & 9)

4. The marking of an interior lattice point is not determined by the geometry of the surface. (eg 2 & 5)

5. The Euler number of an irreducible component of the exceptional divisor is not bounded by 6 from above.



VS.



Thank you for your attention!